Lecture 30

11.10 - Taylor and Maclauren Series

In the previous lecture, we used geometric series and differentiation and integration to find representations of functions as power series...

However, only a small class of functions can be found in this way (e.g., we can't do this for $\sin x$ or e^x). So, what do we do more generally?

Def: We say $f(x)$ has a power series expansion at a if

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a|<R$$

for some $R>0$.

So, this leaves us with some questions to ask:

1) If $f(x)$ has a power series expansion at a, can we tell what it is?

2) For which values of x does $f(x)$ and its power series coincide?

Let's begin by trying to answer the first question.

Taylor Series

Def: If $f(x)$ is a function with infinitely many derivatives at a, the Taylor Series of $f(x)$ about a is the series:

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a=0$, we call it the Maclaurin Series

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Why should we expect this to be the correct power series? Look at derivatives:

If $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Then,

0: $T(a) = f(a) + f'(a)(a-a) + \frac{f''(a)}{2!}(a-a)^2 + \cdots = f(a)$

1: $T'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2 + \cdots = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n}$

$T'(a) = f'(a)$

2: $T''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{2}(x-a)^2 + \cdots = \sum_{n=2}^{\infty} \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n}$

$T''(a) = f''(a)$

and so on...

Thus $T^{(n)}(a) = f^{(n)}(a)$ for all n.

Let's do examples of finding these series:

Ex: Find the Taylor Series at $a=0$ (i.e., the Maclauren Series) for $f(x)=e^x$.

n	0	1	2	3	$\cdots$	n
$f^{(n)}(x)$	e^x	e^x	e^x	e^x	$\cdots$	e^x
$f^{(n)}(0)$	1	1	1	1	$\cdots$	1

Maclauren Series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$

Ex: Find the Taylor Series at $a=0$ for

$$f(x) = \sin x$$

n	0	1	2	3	4	5	6	$\cdots$
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$	$\cdots$
$f^{(n)}(0)$	0	1	0	-1	0	1	0	$\cdots$

$f^{(n)}(0)=0$ for all even n, so let's find a pattern for the odds:

m	0	1	2	3	4	$\cdots$	m
$f^{(2m+1)}(0)$	1	-1	1	-1	1	$\cdots$	$(-1)^m$

Maclauren Series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n \text{ odd}} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}}$

Ex: Using our methods from last class, since

$$(\ln x)' = \frac{1}{x} = \frac{1}{1-[-(x-1)]},$$

we can find a power series representation, centered at $a=1$, of $\ln x$ as:

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Find the Taylor series centered at $a=1$ for $f(x)=\ln x$. How does it compare to the above series?

n	0	1	2	3	4	...	n $(n \geq 1)$
$f^{(n)}(x)$	$\ln x$	$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2!}{x^3}$	$\frac{-3!}{x^4}$		$\frac{(-1)^{n-1}(n-1)!}{x^n}$
$f^{(n)}(1)$	0	1	-1	$2!$	$-3!$		$(-1)^{n-1}(n-1)!$

Taylor Series @ $a=1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{f^{(0)}(1)}{0!}(x-1)^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Reindexing: lower the starting n by 1 & replace n w/ n+1 in the terms.

Theorem: If $f(x)$ has a power series expansion at a, i.e., if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R$$

for some $R > 0$, then that power series is the Taylor series of f at a. In particular, we have that

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

What this theorem is saying is if the function has a power series, then that series is the Taylor series. This is the answer to the first question.

Now, what about the other way? When is $f(x)$ actually equal to its power series? (This is our second question.)

Let $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

This is the <u>n^{th} Taylor Polynomial of f at a</u>. These are the partial sums of the Taylor Series, so for any value of x such that $T_n(x) \to f(x)$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The <u>remainder</u> of the n^{th} Taylor polynomial is

$$R_n(x) = f(x) - T_n(x).$$

The theorem we've been looking for is:

<u>Theorem</u>: Let $f(x)$, $T_n(x)$, and $R_n(x)$ be as above.

If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$,

then f is equal to the sum of its Taylor series on $|x-a| < R$.

How do we compute $\lim_{n\to\infty} R_n(x)$?

Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then $R_n(x)$ satisfies

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \quad \text{for } |x-a| \leq d$$

Ex (Taylor's inequality for e^x) $(a=0)$

Start by finding M: $f^{(n+1)}(x) = e^x$ for any n

So, for $|x| \leq d$, $|f^{(n+1)}(x)| = |e^x| \leq e^d = M$

$$\Rightarrow \boxed{|R_n(x)| \leq \frac{e^d|x|^{n+1}}{(n+1)!}}$$

Ex (Taylor's inequality for $\sin x$) $(a=0)$

$f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$. At any rate, for $|x| \leq d$

$|f^{(n+1)}(x)| \leq 1$

$$\Rightarrow \boxed{|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}}$$

Ex: Prove that e^x is equal to its Maclauren series for all x.

The Taylor inequality is: $|R_n(x)| \leq \frac{e^d |x|^{n+1}}{(n+1)!}$ for $|x| \leq d$.

Now, for any x:

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{e^d |x|^{n+1}}{(n+1)!} = 0 \quad \text{for any } d.$$

Thus $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Ex: Compute the limit

$$\lim_{x\to 0} \frac{\cos(x^5)-1}{x^{10}}$$

The Maclauren series for $\cos(x)$ is: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

So, $\frac{\cos(x^5)-1}{x^{10}} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{10n}}{(2n)!} - 1}{x^{10}} = \frac{1}{x^{10}} \sum_{n=1}^{\infty} \frac{(-1)^n x^{10n}}{(2n)!}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{10n-10}}{(2n)!} = \frac{-1}{2} + \frac{x^{10}}{4!} - \frac{x^{20}}{6!} + \cdots$$

$$\Rightarrow \lim_{x\to 0} \frac{\cos(x^5)-1}{x^{10}} = \lim_{x\to 0}\left(\frac{-1}{2} + \frac{x^{10}}{4!} - \frac{x^{20}}{6!} + \cdots\right) = \boxed{\frac{-1}{2}}$$